

## Efficient Dynamic Models for Flexible Robots

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### Abstract

Dynamic equations of motion of flexible manipulators are more complicated than those of rigid manipulators. The number of equations of motion increases as the number of modes to be included increases. It is difficult to understand the effect of flexible motion on rigid motion via recursive forms of the equations of motion for multi-link arm even if it were efficient. On the other hand, the closed form of the equations of motion is useful in understanding the characteristics of model parameters. However, the equations resulting from existing closed forms are too complex to serve this purpose. Therefore, a method which is structurally well organized and computationally efficient is developed.

### 1 Introduction

One of the primary concerns in manipulator dynamics is computational efficiency. For the efficient form of the manipulator dynamic equations, various recursive formulations for rigid manipulators using Lagrangian [6], Newton-Euler [10], or Kane's method [4], have been proposed. For flexible manipulators, Book used the method of homogeneous transformation matrices. He first considered small linear motions of a massless elastic chain [2] and later considered distributed mass and elasticity [3]. When the recursive formulation is used, the structure of the dynamic model, which is quite useful in providing insight for designing the controller, is destroyed. To overcome this problem, several programs for rigid manipulators have been developed to derive the equations of motion in symbolic form. Symbolic formulation has the advantage of allowing the identification of the distinct components of the model. Maizza-Neto [11] derived symbolically the equations of motion of a two link flexible manipulator by hand. A systematic method to symbolically derive the nonlinear dynamic equations of multi-link flexible manipulators was presented by Cetinkunt [5]. However, he did not explore the structure of the terms in the flexible

manipulator model. The conceptual framework leads to design guidelines for simplifying and reducing the nonlinear kinematic and dynamic coupling of robot dynamics. The physical interpretations and structural characteristics of the Lagrangian dynamic rigid manipulator model was drawn by Tourassis and Neuman [13,14]. The mass matrix is deduced from the masses and center of gravity of links. In turn, the centrifugal and Coriolis coefficients are derived from an inertia matrix through the Christoffel symbol. However, the method of deriving mass matrices is not efficient. Asada [1] presented a method which uses the Jacobian matrix to derive the mass and gravity matrices. This method is found in this paper to be very efficient in the modelling of a flexible manipulator. Low [9] used the Jacobian matrix in deriving the equations of motion of a flexible manipulator. However, the link deformation was not represented in the assumed mode method and the structure of centrifugal and Coriolis force was still complicated and hard to understand.

In this paper, a Lagrangian method is used to derive the equations of motion for a flexible manipulator. The Jacobian matrix is used to derive the mass and gravity matrices. The Coriolis and centrifugal coefficients are derived from the mass matrices using the Christoffel symbol.

### 2 Derivation of Equations of Motion

The total kinetic energy of an elastic link can be written as

$$T = \frac{1}{2} \sum_{i=1}^n \int_0^{l_i} \dot{\mathbf{r}}_i^T \dot{\mathbf{r}}_j \rho_i A_i dx_i \quad (1)$$

where  $\dot{\mathbf{r}}$  is the velocity vector of any point on the elastic link and  $\rho_i$ ,  $A_i$ ,  $l_i$  are the density, the area, and the length of link  $i$  respectively. The velocity vector can be expressed by Jacobian matrix and generalized coordinates.

$$\dot{\mathbf{r}}_i = \mathbf{J}_i \dot{\mathbf{q}} \quad (2)$$

Substitute (2) into (1),

$$\begin{aligned} T &= \frac{1}{2} \sum_{i=1}^n \int_0^{l_i} (J_i \dot{q}_i)^T (J_j \dot{q}_j) \rho_i A_i dx_i \\ &= \frac{1}{2} \sum_{i=1}^n \dot{q}_i^T \left( \int_0^{l_i} J_i^T J_j \rho_i A_i dx_i \right) \dot{q}_j \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n M_{ij} \dot{q}_i \dot{q}_j \end{aligned} \quad (3)$$

where

$$M_{ij} = \int_0^{l_i} J_i^T J_j \rho_i A_i dx_i \quad (4)$$

The potential energy due to gravity is

$$\begin{aligned} U_g &= \sum_{j=1}^n \int_0^{l_j} g^T r_j \rho_j A_j dx_j \\ &= \sum_{j=1}^n m_j g^T r_j \end{aligned} \quad (5)$$

where  $g$  is the  $3 \times 1$  gravity vector and

$$m_j = \int_0^{l_j} \rho_j A_j dx_j \quad (6)$$

The potential energy due to elastic deformation is :

$$U_e = \frac{1}{2} \sum_{i=1}^n \int_0^{l_i} E_i I_i \frac{\partial^2 u_j}{\partial x_j^2} \frac{\partial^2 u_k}{\partial x_k^2} dx_i \quad (7)$$

where  $E$  is Young's modulus of elasticity, and  $I$  is the area moment of inertia.  $u$  is the elastic deflection which can be expressed as follows.

$$u_i = \sum_{j=1}^m \psi_{ij} \delta_{ij} \quad (8)$$

Therefore, the elastic energy can be rewritten as (9)

$$\begin{aligned} U_e &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^m \int_0^{l_i} E_i I_i \frac{\partial^2 \psi_j}{\partial x_j^2} \frac{\partial^2 \psi_k}{\partial x_k^2} \delta_{ij} \delta_{ik} dx_i \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^m K_{ijk} \delta_{ij} \delta_{ik} \end{aligned} \quad (9)$$

where

$$K_{ijk} = \int_0^{l_i} E_i I_i \frac{\partial^2 \psi_j}{\partial x_j^2} \frac{\partial^2 \psi_k}{\partial x_k^2} dx_i \quad (10)$$

Lagrange's equation is

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial (U_g + U_e)}{\partial q_i} = Q_i \quad (11)$$

Substitute the kinetic energy (3) into (11),

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) = \frac{d}{dt} \left( \sum_{j=1}^n M_{ij} \dot{q}_j \right) = \sum_{j=1}^n M_{ij} \ddot{q}_j + \sum_{j=1}^n \frac{dM_{ij}}{dt} \dot{q}_j \quad (12)$$

where

$$\frac{dM_{ij}}{dt} = \sum_{k=1}^n \frac{\partial M_{ij}}{\partial q_k} \frac{dq_k}{dt} = \sum_{k=1}^n \frac{\partial M_{ij}}{\partial q_k} \dot{q}_k \quad (13)$$

Therefore,

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) &= \sum_{j=1}^n M_{ij} \ddot{q}_j + \sum_{j=1}^n \sum_{k=1}^n \frac{\partial M_{ij}}{\partial q_k} \dot{q}_j \dot{q}_k \\ &= \sum_{j=1}^n M_{ij} \ddot{q}_j + \sum_{j=1}^n \sum_{k=1}^n \frac{1}{2} \left( \frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j} \right) \dot{q}_j \dot{q}_k \end{aligned} \quad (14)$$

$$\begin{aligned} \frac{\partial T}{\partial q_i} &= \frac{\partial}{\partial q_i} \left( \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n M_{jk} \dot{q}_j \dot{q}_k \right) \\ &= \sum_{j=1}^n \sum_{k=1}^n \frac{1}{2} \frac{\partial M_{jk}}{\partial q_i} \dot{q}_j \dot{q}_k \end{aligned} \quad (15)$$

Substitute the potential energy (7), (9) to (11),

$$\begin{aligned} \frac{\partial U_g}{\partial q_i} &= \sum_{j=1}^n m_j g^T \frac{\partial r_j}{\partial q_i} \\ &= \sum_{j=1}^n m_j g^T J_j^{(i)} \end{aligned} \quad (16)$$

where  $J_j^{(i)}$  is the  $i$ th column of Jacobian matrix  $J_j$ .

$$\begin{aligned} \frac{\partial U_e}{\partial q_i} &= \frac{\partial}{\partial \delta_{ij}} \left( \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^m K_{ijk} \delta_{ij} \delta_{ik} \right) \\ &= \sum_{j=1}^m K_{ijj} \delta_{ij} \end{aligned} \quad (17)$$

The Lagrangian equations of motion can be written symbolically as follows.

$$\sum_{j=1}^n M_{ij} \ddot{q}_j + \sum_{j=1}^m K_{ijj} \delta_{ij} + \dots \quad (18)$$

$$\sum_{j=1}^n \sum_{k=1}^n \frac{1}{2} \left( \frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{jk}}{\partial q_i} \right) \dot{q}_j \dot{q}_k + \sum_{j=1}^n m_j g^T J_j^{(i)} = \tau_i \quad (9)$$

or

$$\sum_{j=1}^n M_{ij} \ddot{q}_j + \sum_{j=1}^n K_{ij} q_j + \sum_{j=1}^n \sum_{k=1}^n C_{jk}(i) \dot{q}_j \dot{q}_k + G_i g = \tau_i \quad (19)$$

where  $q$  is the vector of generalized coordinates,  $M$  is the generalized mass matrix,  $K$  is the elastic stiffness matrix,  $C$  is the coefficient matrix of Coriolis and centrifugal force,  $G$  is the gravity force,  $\tau$  is the vector of generalized forces.

### 3 Illustrative Example

In this section, equations of motion of a planar two degree of freedom flexible robot are derived as an illustration. In the conventional two serial link robot, there is a difficulty in measuring the end point slope  $\alpha$  of link AB as shown in Fig. 1.a. In order to overcome this problem, the flexible robot with a parallel link mechanism is developed as shown in Fig. 1.b. The angles  $\theta_2$  and  $\theta_3$  are equal because link AD and link BC remain parallel. In this paper, equations of motion of only link AB and link BC are derived because those of the other link can be derived similarly.

#### 3.1 Mass Matrices and Gravity Vectors

Deformed position vectors of each link in Fig. 2.a and 2.b are described as follows:

$$\bar{r}_1 = (x_1 \cos \theta_1 - u_1 \sin \theta_1) i + (x_1 \sin \theta_1 + u_1 \cos \theta_1) j \quad (20)$$

$$\begin{aligned} \bar{r}_2 = & [l_1 \cos \theta_1 - u_{1e} \sin \theta_1 + x_2 \cos(\theta_1 + \theta_2) - u_2 \sin(\theta_1 + \theta_2)] i \\ & + [l_1 \sin \theta_1 + u_{1e} \cos \theta_1 + x_2 \sin(\theta_1 + \theta_2) + u_2 \cos(\theta_1 + \theta_2)] j \end{aligned} \quad (21)$$

where  $i$  and  $j$  are unit vectors along the inertial frame,  $X_0$  and  $Y_0$ . The elastic deformation,  $u_i$ , can be expressed by finite series of mode shape functions which satisfy assumed boundary conditions multiplied by time dependent general coordinates. Suppose that the amplitude of the higher modes is relatively small compared with the first mode, two modes per link are considered in this model.

$$u_1(x_1, t) = \psi_{11}(x_1) \xi_{11}(t) + \psi_{12}(x_1) \xi_{12}(t) \quad (22)$$

$$u_2(x_2, t) = \psi_{21}(x_2) \xi_{21}(t) + \psi_{22}(x_2) \xi_{22}(t) \quad (23)$$

The elastic displacement of the end point is

$$u_{1e} = u_1(l_1, t) \quad (24)$$

Velocity vectors are related to general coordinates by the Jacobian matrix [1].

$$\dot{\bar{r}}_1 = J_1 \dot{q}_{12} \quad (25)$$

$$\dot{\bar{r}}_2 = J_2 \dot{q}_{12} \quad (26)$$

where

$$q_{12} = \{\theta_1, \theta_2, \xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}\}^T \quad (27)$$

$$J_1 = \begin{bmatrix} -u_1 C_1 - x_1 S_1 & 0 & -\psi_{11} S_1 & -\psi_{12} C_1 & 0 & 0 \\ -u_1 S_1 - x_1 C_1 & 0 & \psi_{11} C_1 & \psi_{12} S_1 & 0 & 0 \end{bmatrix} \quad (28)$$

$$J_2 = \begin{bmatrix} -l_1 S_1 - u_{1e} C_1 - u_2 C_{12} - x_2 S_{12} & -u_2 C_{12} - x_2 S_{12} \\ +l_1 C_1 - u_{1e} S_1 - u_2 S_{12} + x_2 C_{12} & -u_2 S_{12} + x_2 C_{12} \\ -\psi_{11e} S_1 & -\psi_{12e} S_1 & -\psi_{21} S_{12} & -\psi_{22} S_{12} \\ \psi_{11e} C_1 & \psi_{12e} C_1 & \psi_{21} C_{12} & \psi_{22} C_{12} \end{bmatrix} \quad (29)$$

The Jacobian matrix,  $J_1$  and  $J_2$ , can be easily derived by the MJac function of SMP(Symbolic Manipulation Program) [12]. Using the Jacobian matrix, mass matrices and gravity vectors are calculated by the following equations:

$$M_{ij} = \sum_{i=1}^2 \int_0^{l_i} \rho_i A_i J_i^T J_i dx_i \quad (30)$$

$$\{G_i\} = \sum_{j=i}^2 \int_0^{l_j} \rho_j A_j J_j [2, i] dx_j \quad (i = 1, 2) \quad (31)$$

The second row of  $J_i$  is used in the gravity vector since the gravity is acting in the negative direction of  $Y_0$ .

Elements of mass matrices and gravity forces are:

$$\begin{aligned} M_{11} = & \int_0^{l_1} (x_1^2 + u_1^2) \rho_1 A_1 dx_1 \\ & - \int_0^{l_2} [l_1^2 + u_{1e}^2 + u_2^2 + x_2^2 \\ & - 2(l_1 x_2 C_2 - l_1 u_2 S_2 \\ & - u_{1e} u_2 C_2 + u_{1e} x_2 S_2)] \rho_2 A_2 dx_2 \end{aligned}$$

$$\begin{aligned} M_{12} = & \int_0^{l_2} (x_2^2 + u_2^2 \\ & - l_1 x_2 C_2 - l_1 u_2 S_2 - u_{1e} u_2 C_2 + u_{1e} x_2 S_2) \rho_2 A_2 dx_2 \end{aligned}$$

$$M_{13} = \int_0^{l_1} x_1 \psi_{11} \rho_1 A_1 dx_1$$

$$+ \psi_{11e} \int_0^{l_2} (l_1 - x_2 C_2 - u_2 S_2) \rho_2 A_2 dx_2$$

$$M_{14} = \int_0^{l_1} x_1 \psi_{12} \rho_1 A_1 dx_1$$

$$+ \psi_{12e} \int_0^{l_2} (l_1 - x_2 C_2 - u_2 S_2) \rho_2 A_2 dx_2$$

$$M_{15} = \int_0^{l_2} \psi_{21} (x_2 + l_1 C_2 + u_{1e} S_2) \rho_2 A_2 dx_2$$

$$M_{16} = \int_0^{l_2} \psi_{22} (x_2 + l_1 C_2 + u_{1e} S_2) \rho_2 A_2 dx_2$$

$$M_{22} = \int_0^{l_2} (x_2^2 + u_2^2) \rho_2 A_2 dx_2$$

$$M_{23} = \psi_{11e} \int_0^{l_2} (x_2 C_2 - u_2 S_2) \rho_2 A_2 dx_2$$

$$\begin{aligned}
M_{24} &= \psi_{12e} \int_0^{l_2} (x_2 C_2 - u_2 S_2) \rho_2 A_2 dx_2 \\
M_{25} &= \int_0^{l_2} x_2 \psi_{21} \rho_2 A_2 dx_2 \\
M_{26} &= \int_0^{l_2} x_2 \psi_{22} \rho_2 A_2 dx_2
\end{aligned} \quad (32)$$

$$M_{33} = \int_0^{l_1} \psi_{11}^2 \rho_1 A_1 dx_1 + \psi_{11e}^2 \int_0^{l_2} \rho_2 A_2 dx_2$$

$$M_{34} = \int_0^{l_1} \psi_{11} \psi_{12} \rho_1 A_1 dx_1 + \psi_{11e} \psi_{12e} \int_0^{l_2} \rho_2 A_2 dx_2$$

$$M_{35} = \psi_{11e} C_2 \int_0^{l_2} \psi_{21} \rho_2 A_2 dx_2$$

$$M_{36} = \psi_{11e} C_2 \int_0^{l_2} \psi_{22} \rho_2 A_2 dx_2$$

$$M_{44} = \int_0^{l_1} \psi_{12}^2 \rho_1 A_1 dx_1 + \psi_{12e}^2 \int_0^{l_2} \rho_2 A_2 dx_2$$

$$M_{45} = \psi_{12e} C_2 \int_0^{l_2} \psi_{21} \rho_2 A_2 dx_2$$

$$M_{46} = \psi_{12e} C_2 \int_0^{l_2} \psi_{22} \rho_2 A_2 dx_2$$

$$M_{55} = \int_0^{l_2} \psi_{21}^2 \rho_2 A_2 dx_2$$

$$M_{56} = \int_0^{l_2} \psi_{21} \psi_{22} \rho_2 A_2 dx_2$$

$$M_{66} = \int_0^{l_2} \psi_{22}^2 \rho_2 A_2 dx_2$$

$$G_1 = \int_0^{l_1} (x_1 C_1 - u_1 S_1) \rho_1 A_1 dx_1$$

$$+ \int_0^{l_2} (l_1 C_1 - u_{1e} S_1 + x_2 C_{12} - u_2 S_{12}) \rho_2 A_2 dx_2$$

$$G_2 = \int_0^{l_2} (x_2 C_{12} - u_2 S_{12}) \rho_2 A_2 dx_2 \quad (33)$$

where

$$\psi_{ije} = \psi_{ij}(l_i)$$

The integrals in equations are labeled as follows.

$$m_i = \int_0^{l_i} \rho_i A_i dx_i \quad (34)$$

$$m_i l_{ic} = \int_0^{l_i} x_i \rho_i A_i dx_i \quad (35)$$

$$J_i = \int_0^{l_i} x_i^2 \rho_i A_i dx_i \quad (36)$$

$$LM_{ij} = \int_0^{l_i} \psi_{ij}(x_i) \rho_i A_i dx_i \quad (37)$$

$$AM_{ij} = \int_0^{l_i} x_i \psi_{ij}(x_i) \rho_i A_i dx_i \quad (38)$$

$$NM_{ij} = \int_0^{l_i} \psi_{ij}^2(x_i) \rho_i A_i dx_i \quad (39)$$

where  $l_{ic}$  is center of mass of link  $i$ .

The first three terms are parameters which are related to a rigid motion. These are called zeroth, first, and second moments of inertia respectively. On the other hand, the last three terms are parameters which are related to a flexible motion.  $LM_{ij}$  and  $AM_{ij}$  are called the modal momentum coefficients and the modal angular momentum coefficients respectively [7]. The physical meaning of these terms is not easy to explain. However, these are have the following properties [7].

$$\sum_{j=1}^{\infty} LM_j^2 = m \quad (40)$$

$$\sum_{j=1}^{\infty} LM_j AM_j = m l_c \quad (41)$$

$$\sum_{j=1}^{\infty} AM_j^2 = J \quad (42)$$

$NM_{ij}$  are used for the normalization of mode shape functions. Generally, these coefficients have been chosen equal to 1 or the total moment of inertia of the link.

### 3.2 Centrifugal and Coriolis force

The velocity coupling matrix which are consist of coefficients of centrifugal and Coriolis force can be derived from the mass matrix using the Christoffel symbol [13,14].

$$C_{jk}(i) = \frac{1}{2} \left\{ \frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j} - \frac{\partial M_{jk}}{\partial q_i} \right\} \quad (43)$$

$C_{jk}(i)$  characterizes the effects on link  $i$  which are caused by the coupled velocities of link  $j$  and  $k$ . The diagonal elements for  $j = k$  are the coefficients of the centrifugal force. The off-diagonal elements for  $j \neq k$  are the coefficients of the Coriolis force.

In the flexible arm dynamics, the states can be partitioned into the rigid states  $\theta$  and the flexible states  $\delta$ .

$$\sum_{i=1}^2 A_{ij} \ddot{\theta}_j + \sum_{j=3}^6 B_{ij} \ddot{\delta}_j + \sum_{j=1}^2 \sum_{k=1}^2 P_{jk}(i) \dot{\theta}_j \dot{\theta}_k + \sum_{j=1}^2 \sum_{k=3}^6 Q_{jk}(i) \dot{\theta}_j \dot{\delta}_k$$

$$+ \sum_{j=3}^6 \sum_{k=3}^6 R_{jk}(i) \dot{\delta}_j \dot{\delta}_k + G_i = \tau_i \quad (i = 1, 2) \quad (44)$$

$$\begin{aligned} & \sum_{i=1}^2 B_{ji} \ddot{\theta}_j + \sum_{j=3}^6 D_{ij} \ddot{\delta}_j + \sum_{j=1}^2 \sum_{k=1}^2 \hat{P}_{jk}(i) \dot{\theta}_j \dot{\theta}_k + \sum_{j=1}^2 \sum_{k=3}^6 \hat{Q}_{jk}(i) \dot{\theta}_j \dot{\delta}_k \\ & + \sum_{j=3}^6 \sum_{k=3}^6 \hat{R}_{jk}(i) \dot{\delta}_j \dot{\delta}_k + \sum_{j=1}^2 K_{ij} \delta_j = 0 \quad (i = 3, 6) \quad (45) \end{aligned}$$

Therefore, each velocity coupling matrix can be written as follows:

$$P_{jk}(i) = \frac{1}{2} \left\{ \frac{\partial A_{ij}}{\partial q_k} + \frac{\partial A_{ik}}{\partial q_j} - \frac{\partial A_{jk}}{\partial q_i} \right\} \quad (46)$$

$$Q_{jk}(i) = \frac{1}{2} \left\{ \frac{\partial A_{ij}}{\partial q_k} + \frac{\partial B_{ik}}{\partial q_j} - \frac{\partial B_{jk}}{\partial q_i} \right\} \quad (47)$$

$$R_{jk}(i) = \frac{1}{2} \left\{ \frac{\partial B_{ij}}{\partial q_k} + \frac{\partial B_{ik}}{\partial q_j} - \frac{\partial D_{jk}}{\partial q_i} \right\} \quad (48)$$

$$\hat{P}_{jk}(i) = \frac{1}{2} \left\{ \frac{\partial B_{ij}}{\partial q_k} + \frac{\partial B_{ik}}{\partial q_j} - \frac{\partial A_{jk}}{\partial q_i} \right\} \quad (49)$$

$$\hat{Q}_{jk}(i) = \frac{1}{2} \left\{ \frac{\partial B_{ij}}{\partial q_k} + \frac{\partial D_{ik}}{\partial q_j} - \frac{\partial B_{jk}}{\partial q_i} \right\} \quad (50)$$

$$\hat{R}_{jk}(i) = \frac{1}{2} \left\{ \frac{\partial D_{ij}}{\partial q_k} + \frac{\partial D_{ik}}{\partial q_j} - \frac{\partial D_{jk}}{\partial q_i} \right\} \quad (51)$$

Because mass submatrix  $D_{ij}$  are not the function of elastic state  $\delta_i$  in equation (29),  $\hat{R}_{jk}(i)$  is eliminated. The number of independent centrifugal and Coriolis coefficients also can be reduced using the symmetry and the reflective coupling properties [13,14].

$$C_{jk}(i) = C_{kj}(i) \quad (52)$$

$$C_{jk}(i) = -C_{ji}(k) \quad \text{for } j \leq i, k \quad (53)$$

The reflective coupling property that Tourassis and Neuman finds for rigid arms is not always valid in the flexible case. Therefore, even though the symbolic manipulation program can be used as the computational tool, the simplification procedure must be completed under the supervision of the analyst.

Using these properties, the following independent terms are drawn from elements of the velocity coupling matrix  $C_{jk}(i)$ .

$$d_1 = \int_0^{l_2} \{ (u_{1e} C_2 - l_1 S_2) x_2 - (u_{1e} S_2 + l_1 C_2) u_2 \rho_2 A_2 dx_2 \}$$

$$d_{21} = \int_0^{l_1} \psi_{11}^2 q_{11} \rho_1 A_1 dx_1$$

$$+ \psi_{11e} \left[ \int_0^{l_2} (S_2 x_2 + C_2 u_2 + u_{1e}) \rho_2 A_2 dx_2 \right]$$

$$d_{22} = \int_0^{l_1} \psi_{12}^2 q_{12} \rho_1 A_1 dx_1$$

$$+ \psi_{12e} \left[ \int_0^{l_2} (S_2 x_2 + C_2 u_2 + u_{1e}) \rho_2 A_2 dx_2 \right]$$

$$d_{31} = \psi_{11e} \int_0^{l_2} (S_2 x_2 + C_2 u_2) \rho_2 A_2 dx_2$$

$$d_{32} = \psi_{12e} \int_0^{l_2} (S_2 x_2 + C_2 u_2) \rho_2 A_2 dx_2$$

$$d_{41} = \int_0^{l_2} \psi_{21} (\psi_{21} q_{21} + \psi_{22} q_{22}) \rho_2 A_2 dx_2$$

$$d_{42} = \int_0^{l_2} \psi_{22} (\psi_{21} q_{21} + \psi_{22} q_{22}) \rho_2 A_2 dx_2 \quad (54)$$

$$d_{51} = \int_0^{l_2} \psi_{21} \{ (\psi_{21} q_{21} + \psi_{22} q_{22}) + (u_{1e} C_2 - l_1 S_2) \} \rho_2 A_2 dx_2$$

$$d_{52} = \int_0^{l_2} \psi_{22} \{ (\psi_{21} q_{21} + \psi_{22} q_{22}) + (u_{1e} C_2 - l_1 S_2) \} \rho_2 A_2 dx_2$$

$$d_{61} = \psi_{11e} S_2 \int_0^{l_2} \psi_{21} \rho_2 A_2 dx_2$$

$$d_{62} = \psi_{11e} S_2 \int_0^{l_2} \psi_{22} \rho_2 A_2 dx_2$$

$$d_{71} = \psi_{12e} S_2 \int_0^{l_2} \psi_{21} \rho_2 A_2 dx_2$$

$$d_{72} = \psi_{12e} S_2 \int_0^{l_2} \psi_{22} \rho_2 A_2 dx_2$$

Using these coefficients, the velocity coupling matrix for the two link example can be simplified as follows:

$$C(1) = \begin{bmatrix} 0 & d_1 & d_{21} & d_{22} & d_{41} & d_{42} \\ & d_1 & 0 & 0 & d_{41} & d_{42} \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{bmatrix} \quad (55)$$

$$C(2) = \begin{bmatrix} -d_1 & 0 & d_{31} & d_{32} & d_{51} & d_{52} \\ & 0 & 0 & 0 & d_{51} & d_{52} \\ & & 0 & 0 & -d_{61}/2 & -d_{62}/2 \\ & & & 0 & -d_{71}/2 & -d_{72}/2 \\ & & & & 0 & 0 \\ & & & & & 0 \end{bmatrix} \quad (56)$$

## References

- $$C(3) = \begin{bmatrix} -d_{21} & -d_{31} & 0 & 0 & -d_{61} & -d_{62} \\ & -d_{31} & 0 & 0 & -d_{61}/2 & -d_{62}/2 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{bmatrix} \quad (57)$$
- $$C(4) = \begin{bmatrix} -d_{22} & -d_{32} & 0 & 0 & -d_{71} & -d_{72} \\ & -d_{32} & 0 & 0 & -d_{71}/2 & -d_{72}/2 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{bmatrix} \quad (58)$$
- $$C(5) = \begin{bmatrix} -d_{41} & -d_{51} & d_{61} & d_{71} & 0 & 0 \\ & -d_{51} & d_{61}/2 & d_{71}/2 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{bmatrix} \quad (59)$$
- $$C(6) = \begin{bmatrix} -d_{42} & -d_{52} & d_{62} & d_{72} & 0 & 0 \\ & -d_{52} & d_{62}/2 & d_{72}/2 & 0 & 0 \\ & & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{bmatrix} \quad (60)$$
- ## 4 Conclusion
- Mass matrices and gravity vectors are directly derived from the Jacobian matrices which are easily calculated from position vectors by SMP. Because the deriving procedure is simple, it reduces the possibility of producing the incorrect equations. Furthermore, this form can easily expand the model to higher modes expanding elastic deformations as series of mode shape functions. The coefficients of centrifugal and Coriolis force are derived from the mass matrices by Christoffel symbol and are simplified by using several structural properties. The resulting velocity coupling matrices have a structure which is useful to reduce the number of terms calculated, to check correctness, or to extend the model to higher order. Some procedures for deriving the velocity coupling are not computerized. In the future, an even more systematic derivation method may be possible.
- [1] Asada, H., and Slotine, J.-J.E., "Robot Analysis and Control," John Wiley & Sons, Inc, 1986.
  - [2] Book, W.J., "Analysis of Massless Elastic Chains With Servo Controlled Joints," *ASME Journal of Dynamic Systems, Measurement, and Control*, 101(3), Sep. 1979, pp 187-192
  - [3] Book, W.J., "Recursive Lagrangian Dynamics of Flexible Manipulator Arms," *The International Journal of Robotics Research*, 3(3), Fall 1984, pp 87-101.
  - [4] Burdick, J., "An Algorithm for Generation of Efficient Manipulator Dynamic Equations," *Proc. IEEE International Conf. on Robotics and Automation*, San Francisco, CA, 1986, pp 212-218.
  - [5] Cetinkunt, S., "Symbolic Modelling and Dynamic Analysis of Flexible Manipulators," *Proc. IEEE International Conf. on Robotics and Automation*, Raleigh, NC, 1987, pp -
  - [6] Hollerbach, J.M., "A Recursive Lagrangian Formulation of Manipulator Dynamics and a Comparative Study of Dynamic Formulation Complexity," *IEEE Trans. Systems, Man, and Cybernetics*, 10(11), Nov. 1980, pp 730-736.
  - [7] Hughes, P.C., "Modal Identities for Elastic Bodies with Application to Vehicle Dynamics and Control," *Journal of Applied Mechanics*, 47(1), Mar. 1980, pp 177-184.
  - [8] Kane, T.R., and Levinson, D.A., "The Use of Kane's Dynamical Equations in Robotics," *The International Journal of Robotics Research*, 2(3), Fall 1983, pp 3-20.
  - [9] Low, K.H., "A Systematic Formulation of Dynamic Equations for Robot Manipulators with Elastic link," *Journal of Robotic Systems*, 4(3) 1987, pp 435-456.
  - [10] Luh, J.Y.S., Walker, M.W., and Paul, R.P.C., "On-Line Computational Scheme for Mechanical Manipulators," *ASME Journal of Dynamic Systems, Measurement, and Control*, 102, Jun. 1980, pp 69- 76.
  - [11] Maizza - Neto, O., "Modal Analysis and Control of Flexible Manipulator Arms," PhD Thesis, Dept. of Mechanical Engineering, MIT, 1974.
  - [12] *SMP Reference Manual*, Inference Corporation, 1983

[13] Tourassis, V.D., and Neuman, C.P., "Properties and Structure of Dynamic Robot Models for Control Engineering Applications," *Mechanism and Machine Theory*, 20(1), 1985, pp 27-40.

[14] Tourassis, V.D., and Neuman, C.P., "The Inertial Characteristics of Dynamic Robot Models," *Mechanism and Machine Theory*, 20(1), 1985, pp 41-52.

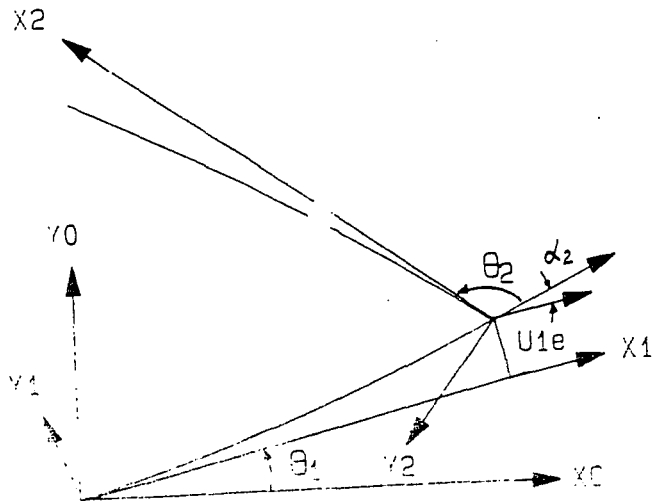


Fig. 1.a Serial Link

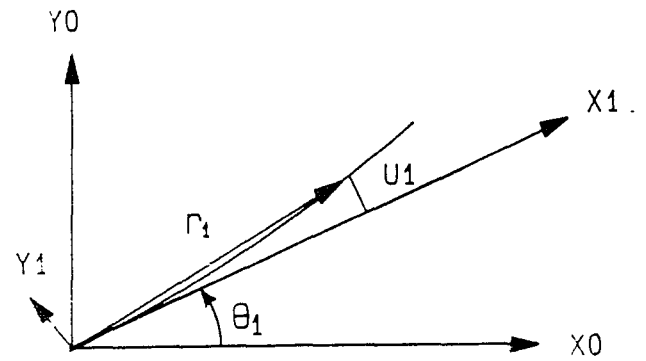


Fig. 2.a Position Vector of Lower Link

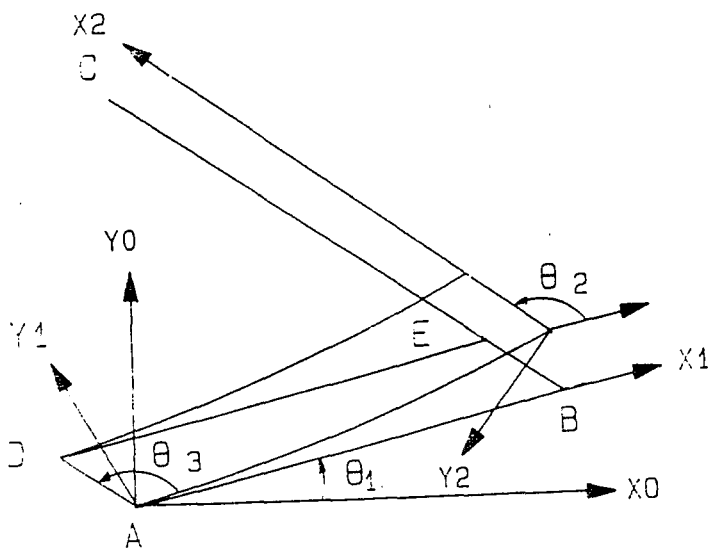


Fig. 1.b Parallel Link

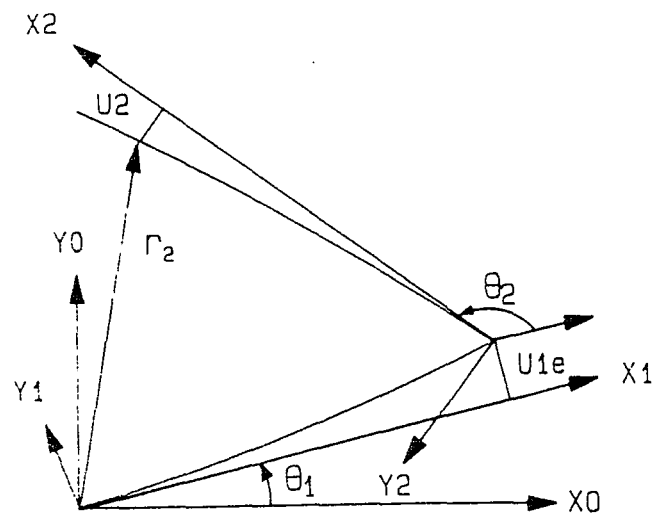


Fig. 2.b Position Vector of Upper Link